UDC 517.938:534.17

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INFLUENCE OF THE DELAY ON THE OCCURRENCE OF DETERMINISTIC CHAOS IN SOME NON-IDEAL PENDULUM SYSTEMS

Background. The influence of the delay of interaction between pendulum and electric motor and the delay of the medium on the dynamics of non-ideal pendulum systems of the type "pendulum-electric motor" is considered. Mathematical model of this system is a system of ordinary differential equations with delay.

Objective. The influence of delay factors on steady-state oscillations of non-ideal pendulum systems of the type "pendulum–electric motor" is studied.

Methods. The approaches that reduce the mathematical model of the system to a system of three or fifteen differential equations without delay are suggested. For general analysis of nonlinear dynamics the maps of dynamical regimes are constructed. These maps allow conducting a qualitative identification of the type of steady-state regime. The construction of dynamical regimes maps is based on analysis and processing of spectrum of Lyapunov characteristic exponents. Phase portraits of regular and chaotic attractors are constructed.

Results. The use of three-dimensional mathematical model to study the dynamics of "pendulum–electric motor" systems is sufficient only at small values of the delay. For relatively high values of the delay multi-dimensional system of fifteen equations should be used.

Conclusions. The essential influence of the delay on qualitative change in the dynamic characteristics in "pendulum–electric motor" systems is shown. In some cases the delay is the controlling factor in the process of chaotization of pendulum systems.

Keywords: pendulum system; systems with limited excitation; maps of dynamical regimes; regular and chaotic attractors; delay factors.

Inroduction

In the majority of studies the dynamics of pendulum systems are being conducted without taking into account the limitations of excitation source power, so it is assumed that the power of excitation source considerably exceeds the power that consumes the vibrating system. Such systems are called ideal in sense of Sommerfeld–Kononenko [1]. In many cases such idealization leads to qualitative and quantitative errors in describing dynamical regimes of pendulum systems.

Modern development of energy efficient and energy-preserving technologies requires the highest minimization of excitation source power of oscillatory systems. This leads to the fact that the energy of excitation source is comparable to the energy consumed by the oscillating system. Such systems as "source of excitation-oscillating subsystem" are called non-ideal by Sommerfeld–Kononenko [1]. In mathematical modeling of such systems, the limitation of excitation source power must be always taken into account.

Another important factor that significantly affects the change of steady-state regimes of dynamical systems is the presence of different in their physical substance, factors of delay. The delay factors are always present in rather extended systems due to the limitations of signal transmission speed: stretching, waves of compression, bending, current strength, etc. In some cases, taking into account factors of delay leads only to minor quantitative changes in dynamic characteristics of pendulum systems. In other cases, taking into account these factors allow to identify qualitative changes in dynamic characteristics

The study of the influence of delay factors on the dynamical stability of equilibrium positions of pendulum systems was initiated by Yu. A. Mitropolsky [2, 3]. Initially only ideal pendulum models were considered.

In this paper the oscillations of non-ideal pendulum systems of the type "pendulum-electric motor" with taking into account various factors of delay are considered. Mathematical models of pendulum system with limited excitation, taking into account the influence of different factors of delay, were first obtained in [4, 5]. The influence of delay factors on existence and dynamic stabilization of pendulum equilibrium positions at limited excitation was studied. Later the existence of chaotic attractors in nonideal systems "pendulum–electric motor" was discovered and proved that the main cause of chaos is limited excitation [6, 7].

Problem statement

The aim of this paper is to study the influence of the delay of interaction between pendulum and electric motor and the delay of the medium in non-ideal pendulum systems of the type "pendulum-electric motor" on steady-state regular and chaotic regimes of oscillations. Construction and analysis of dynamical regimes maps of these systems.

Delay factors in "pendulum–electric motor" system

Mathematical model of the dynamical system "pendulum-electric motor" in the absence of any delay factors can be written as the following system of equations [4, 5, 7]:

$$\begin{cases} \dfrac{dy_1}{d\tau} = Cy_1 - y_2 y_3 - \dfrac{1}{8}(y_1^2 y_2 + y_2^3); \\ \dfrac{dy_2}{d\tau} = Cy_2 + y_1 y_3 + \dfrac{1}{8}(y_1^3 + y_1 y_2^2) + 1; \\ \dfrac{dy_3}{d\tau} = Dy_2 + Ey_3 + F, \end{cases}$$

where phase variables y_1, y_2 – describe the deviation of the pendulum from the vertical and phase variable y_3 – is proportional to the rotation speed of the motor shaft. The system parameters are defined by

$$C = -\delta_1 \varepsilon^{-2/3} \varpi_0^{-1},\ D = -\frac{2ml^2}{I},$$
$$F = \frac{2l^{2/3}}{a^{2/3}}\left(\frac{N_0}{\varpi_0} + E\right),$$

where m – the pendulum mass, l – the reduced pendulum length, ϖ_0 – natural frequency of the pendulum, a – the length of the electric motor crank, $\varepsilon = \frac{a}{l}$, δ_1 – damping coefficient of the medium resistance force, I – the electric motor moment of inertia, E, N_0 – constants of the electric motor static characteristics.

Let us consider the following system of equations:

$$\begin{cases} \dfrac{dy_1(\tau)}{d\tau} = Cy_1(\tau-\delta) - y_2(\tau) y_3(\tau-\gamma) - \\ -\dfrac{1}{8}(y_1^2(\tau) y_2(\tau) + y_2^3(\tau)), \\ \dfrac{dy_2(\tau)}{d\tau} = Cy_2(\tau-\delta) + y_1(\tau) y_3(\tau-\gamma) + \\ +\dfrac{1}{8}(y_1^3(\tau) + y_1(\tau) y_2^2(\tau)) + 1, \\ \dfrac{dy_3(\tau)}{d\tau} = Dy_2(\tau-\gamma) + Ey_3(\tau) + F. \end{cases} \qquad (1)$$

This system is a system of equations with constant delay. Positive constant parameter γ was introduced to account the delay effects of electric motor impulse on the pendulum. We assume that the delay of the electric motor response to the impact of the pendulum inertia force is also equal to γ. Taking into account the delay γ conditioned by the fact that the wave velocity perturbations on the elements of the construction has a finite value that depends on the properties of external fields, for instance, the temperature field. In turn, the constant positive parameter δ characterizes the delay of the medium reaction on the dynamical state of the pendulum. This delay is due to the limited sound velocity in that medium.

Let us consider two approaches that allow reducing the time-delay system (1) to the system of equations without delay. The first approach is as follows. Assuming a small delay, we can write

$$y_i(\tau-\gamma) = y_i(\tau) - \frac{y_i(\tau)}{d\tau}\gamma + \ldots,\quad i = 2, 3;$$
$$y_i(\tau-\delta) = y_i(\tau) - \frac{y_i(\tau)}{d\tau}\delta + \ldots,\quad i = 1, 2.$$

Then, if $C\delta \neq -1$, we get the following system of equations:

$$\begin{cases} \dfrac{dy_1}{d\tau} = \dfrac{1}{1+C\delta}\Big(Cy_1 - y_2[y_3 - \gamma(Dy_2 + Ey_3 + \\ + F)]y_3 - \dfrac{1}{8}(y_1^2 y_2 + y_2^3)\Big); \\ \dfrac{dy_2}{d\tau} = \dfrac{1}{1+C\delta}\Big(Cy_2 + y_1 y_3 - y_1\gamma(Dy_2 + \\ + Ey_3 + F) + \dfrac{1}{8}(y_1^3 + y_1 y_2^2) + 1\Big); \\ \dfrac{dy_3}{d\tau} = (1 - C\gamma)Dy_2 - \dfrac{D\gamma}{8}(y_1^3 + y_1 y_2^2 + \\ + 8y_1 y_2^2 + 8y_1 y_3 + 8) + Ey_3 + F. \end{cases} \qquad (2)$$

The obtained system of equations is already a system of ordinary differential equations. Delays are included in this system as additional parameters.

In order to approximate the system (1) another, more precise, method can be used [8, 9]. In the case of the presence in the system (1) the delay of interaction between pendulum and electric motor γ and the delay of the medium δ, i.e. $\gamma > 0, \delta > 0$, let us divide the segments $[-\gamma; 0]$ and $[-\delta; 0]$ into m equal parts. We introduce the following notation

$$y_1\left(\tau-\frac{i\delta}{m}\right)=y_{1i}(\tau),\ y_2\left(\tau-\frac{i\gamma}{m}\right)=y_{2i}(\tau),$$

$$y_2\left(\tau-\frac{i\delta}{m}\right)=\tilde{y}_{2i}(\tau),\ y_3\left(\tau-\frac{i\gamma}{m}\right)=y_{3i}(\tau),\ i=\overline{0,m}.$$

Then, using difference approximation of derivative [8, 9] we obtain

$$\begin{cases}\dfrac{dy_{10}(\tau)}{d\tau}=Cy_{1m}(\tau)-y_{20}(\tau)y_{3m}(\tau)-\\ -\dfrac{1}{8}(y_{10}^2(\tau)y_{20}(\tau)+y_{20}^3(\tau));\\ \dfrac{dy_{20}(\tau)}{d\tau}=C\tilde{y}_{2m}(\tau)+y_{10}(\tau)y_{3m}(\tau)+\\ +\dfrac{1}{8}(y_{10}^3(\tau)+y_{10}(\tau)y_{20}^2(\tau))+1;\\ \dfrac{dy_{30}(\tau)}{d\tau}=Dy_{2m}(\tau)+Ey_{30}(\tau)+F;\\ \dfrac{dy_{1i}(\tau)}{d\tau}=\dfrac{m}{\delta}(y_{1i-1}(\tau)-y_{1i}(\tau)),\ i=\overline{1,m};\\ \dfrac{dy_{2i}(\tau)}{d\tau}=\dfrac{m}{\gamma}(y_{2i-1}(\tau)-y_{2i}(\tau)),\ i=\overline{1,m};\\ \dfrac{d\tilde{y}_{2i}(\tau)}{d\tau}=\dfrac{m}{\delta}(\tilde{y}_{2i-1}(\tau)-\tilde{y}_{2i}(\tau)),\ i=\overline{1,m};\\ \dfrac{dy_{3i}(\tau)}{d\tau}=\dfrac{m}{\gamma}(y_{3i-1}(\tau)-y_{3i}(\tau)),\ i=\overline{1,m}.\end{cases}\qquad(3)$$

It is a system of ordinary differential equations of $(2m+3)$-th order. As in the system (2), the delays γ and δ are included in these systems as additional parameters. We note that the solutions y_1, y_2, y_3 of the system (1) are described by the functions y_{10}, y_{20}, y_{30} of the system (3).

Choosing a sufficiently large m in the system (3), the system (1) will be very well approximated by the system (3) [8]. In this paper the system of equation (3) was considered at $m=3$, so the system (3) has 15 equations. The calculations of cases $m>3$, with a significant increase the number of equations, were also carried out. It was established, that increasing the number of equations has practically no effect on identification and description of steady-state regimes of “pendulum-electric motor” system. But it significantly increases the complexity of constructing characteristics, which are necessary for study the steady-state regimes of oscillations. Therefore, the use of mathematical model (3) at $m=3$ is optimal for studying the influence of delay on regular and chaotic dynamics of “pendulum-electric motor” system.

Therefore, we obtained three-dimensional (2) and fifteen-dimensional (3) mathematical models each describing the system of equations with delay (1). These models are the systems of nonlinear differential equations, so in general the study of steady-state regimes can be carried out only by using numerical methods and algorithms. The methodology of such studies is described in detail in [7].

Maps of dynamical regimes

For general analysis of nonlinear dynamics the maps of dynamical regimes are constructed. These maps provide a crucial information about the type of steady-state regime of the the system depending on its parameters. The construction of dynamical regimes maps is based on analysis and processing of spectrum of Lyapunov characteristic exponents [7]. Where necessary, for more accurate determination of steady-state regime of the system, we study other characteristics of attractors: Poincare sections and maps, Fourier spectrums, phase portraits and distributions of the invariant measure.

Let us consider the behavior of the systems (2) and (3) when the parameters are $C=-0.1$, $D=-0.6$, $E=-0.44$, $F=0.3$. In Fig. 1 the maps of dynamical regimes are shown. The map in fig. 1, a was built for three-dimensional model (2) and the map in Fig. 1, *b* was built for fifteen-dimensional model (3). These figures illustrate the effect of the delay of interaction between pendulum and electric motor γ and the delay of the medium δ on changing the type of steady-state regime of the systems. The dark-grey areas of the maps correspond to equilibrium positions of the system. The light-grey areas of the maps correspond to limit cycles of the system. And finally, the black areas of the maps correspond to chaotic attractors.

We can notice a certain similarity the maps in Fig. 1 *a*, *b*. At small values of the delays both systems have stable equilibrium positions (dark-grey areas in the figures). With an increase of the delay of the medium δ the type of steady-state regime does not change. It is still remains stable equilibrium positions. However, increasing the delay of interaction between electric motor and pendulum, stable equilibrium position is replaced by the area of limit cycles. Moreover in this area of periodic regimes the area of chaotic attractors is built in. With further increase of the delay γ the attractor of both systems becomes stable equilibrium position.

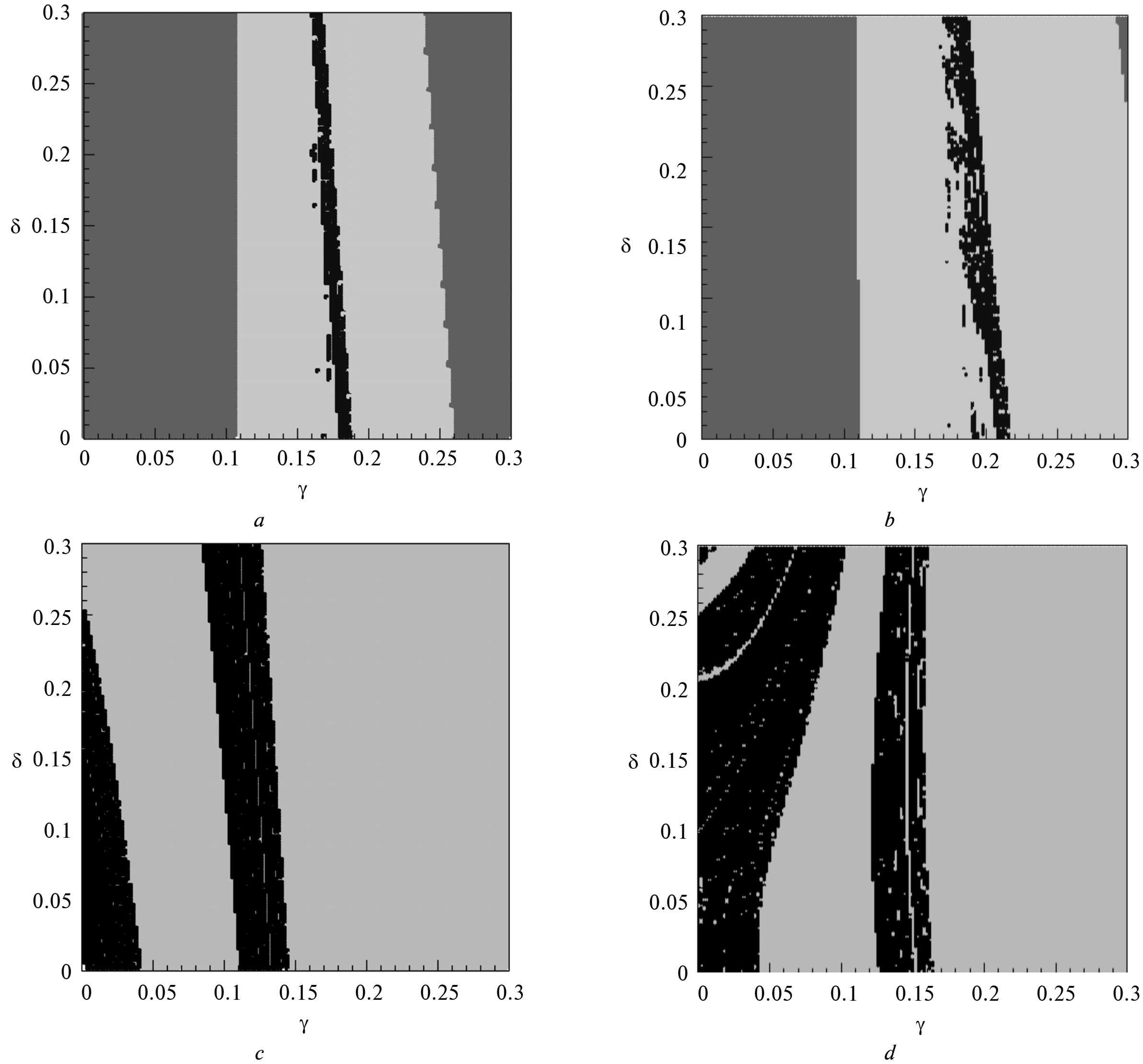

Fig. 1. Maps of dynamical regimes of the system (2) (*a*, *c*) and the system (3) (*b*, *d*)

Let us study the dynamics of the systems (2) and (3) at other values of the parameters. The maps of dynamical regimes of three-dimensional system (2) and fifteen-dimensional system (3) at $C=-0.1, D=-0.53, E=-0.6, F=0.19$ are built respectively in Fig. 1, *c*, *d*. At small values of the delays both systems has chaotic attractors (black areas in the maps). With an increase of the delay of interaction between pendulum and electric motor γ the region of chaos is replaced by the region of periodic regimes. Then again chaos arises in the systems. Further this area is replaced by the area of limit cycles.

The obtained maps of dynamical regimes allow us to conduct a quick qualitative identification of the type of steady-state regime of the systems (2) and (3). On the basis of constructed maps, more detailed studies of emerging dynamical regimes can be carried out. Particularly we can study the transition from regular to chaotic regimes [10, 11].

As seen from the constructed maps of dynamical regimes, the dynamics of three-dimensional system (2) and fifteen-dimensional system (3) is the same only at small values of the delays γ and δ. With an increase of the delays the differences of the dynamics of these systems is very significant.

For instance, when the parameters are $C=-0.1, D=-0.53, E=-0.6, F=0.19$ and the delay of interaction between pendulum and electric motor $\gamma=0.05$ and the delay of the medium

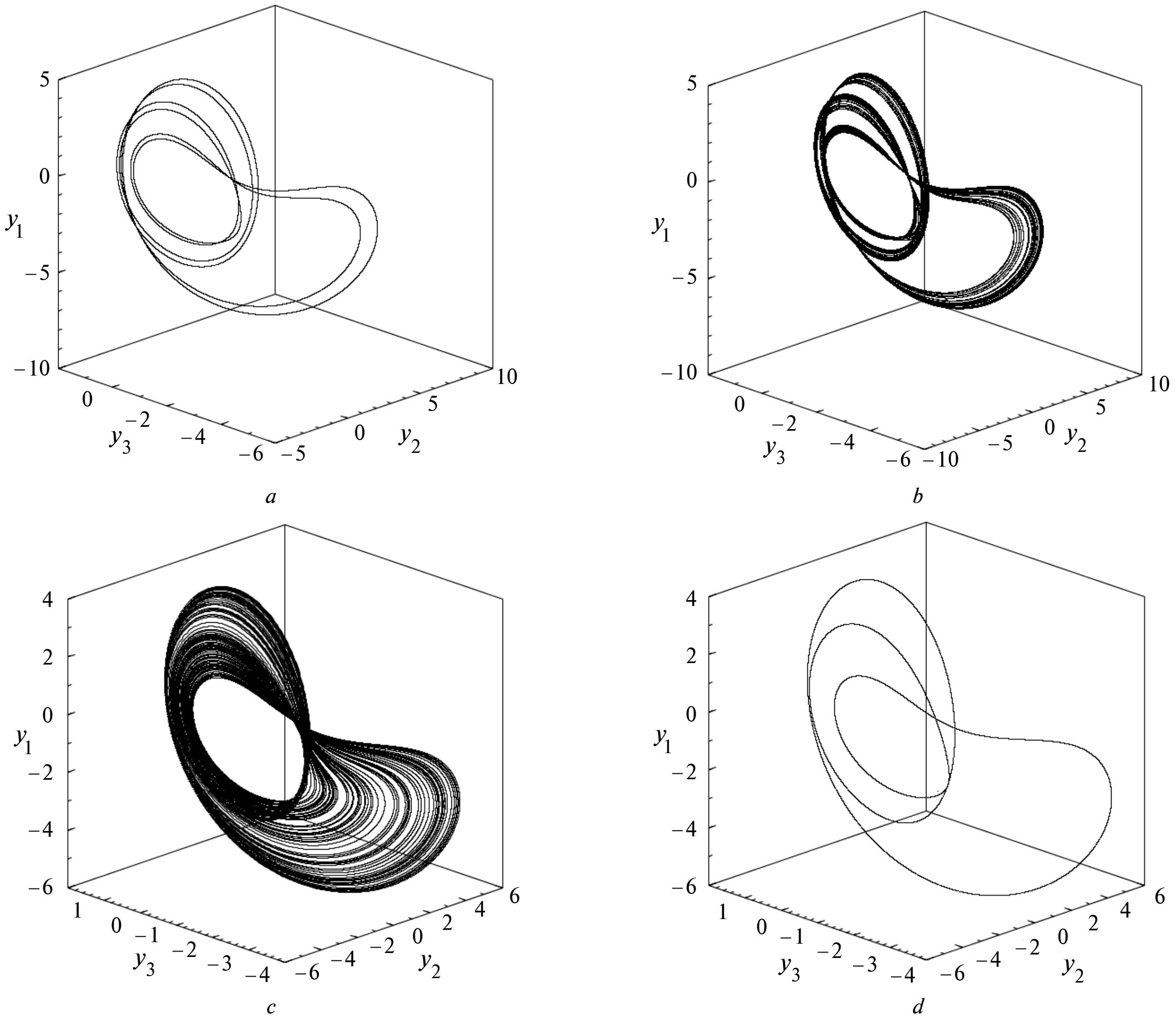

Fig. 2. Phase portraits of attractors of the system (2) (*a*, *c*) and the system (3) (*b*, *d*)

$\delta = 0.1$ the steady-state regime of three-dimensional system (2) is periodic and the attractor is limit cycle. Phase portrait of this attractor is shown in Fig. 2, *a*. Whereas at this values of the parameters and the delays fifteen-dimensional system (3) has steady-state chaotic dynamical regime. Phase portrait of the chaotic attractor of the system (3) is built in Fig. 2, *b*.

It is also possible a different situation. For instance at the delays $\gamma = 0.12, \delta = 0.05$ the system (2) has chaotic steady-state regime of oscillations. Phase portrait of the attractor is shown in fig. 2, *c*. Whereas at this values of the delay fifteen-dimensional system (3) has regular periodic dynamical regime and its attractor is limit cycle (Fig. 2, *d*).

Conclusions

Taking into account various factors of delay in "pendulum–electric motor" systems is crucial. The presence of delay in such systems can affect qualitative change in the dynamic characteristics. In some cases the delay is the main reason of origination as well as vanishing of chaotic attractors. So delay is the controlling factor in the process of chaotization of pendulum systems.

The use of three-dimensional mathematical model to study the dynamics of "pendulum–electric motor" systems is sufficient only at small values of the delay. For relatively high values of the delay multi-dimensional system of fifteen equations should be used.

In future research is planned to study the influence of variable in time delay factors on oscillations of non-ideal dynamical systems of the type "pendulum–electric motor".

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О.Ю. Швець, О.М. Макасєєв

ВПЛИВ ЗАПІЗНЮВАННЯ НА ВИНИКНЕННЯ ДЕТЕРМІНОВАНОГО ХАОСУ В ДЕЯКИХ НЕІДЕАЛЬНИХ МАЯТНИКОВИХ СИСТЕМАХ

Проблематика. Розглядається вплив запізнювання взаємодії між маятником і електродвигуном та запізнювання реакції середовища на динаміку неідеальних маятникових систем типу "маятник–електродвигун". Математичною моделлю такої системи є система звичайних диференціальних рівнянь із запізнюванням.

Мета дослідження. Мета роботи – дослідити вплив факторів запізнювання на усталені режими коливань неідеальних маятникових систем типу "маятник–електродвигун".

Методика реалізації. Запропоновано підходи, які дають змогу звести математичну модель системи до системи трьох або п'ятнадцяти диференціальних рівнянь без запізнювання. Для загального аналізу нелінійної динаміки цих систем побудовано карти динамічних режимів, які дають можливість проводити якісну ідентифікацію типу усталеного режиму. Методика побудови карт динамічних режимів базується на аналізі спектра ляпуновських характеристичних показників. Побудовано фазові портрети регулярних і хаотичних атракторів.

Результати дослідження. Встановлено, що для дослідження динаміки систем "маятник–електродвигун" за малих значень запізнювання достатньо використовувати тривимірну математичну модель. За порівняно великих значень запізнювання необхідно використовувати багатовимірну математичну модель, яка складається з п'ятнадцяти рівнянь.

Висновки. Показано суттєвий вплив запізнювання на якісну зміну динамічних характеристик у системах типу "маятник–електродвигун". У деяких випадках запізнювання є керуючим фактором у процесі хаотизації маятникових систем.

Ключові слова: маятникові системи; системи з обмеженим збудженням; карти динамічних режимів; регулярні та хаотичні атрактори; запізнювання.

А.Ю. Швец, А.М. Макасеев

ВЛИЯНИЕ ЗАПАЗДЫВАНИЯ НА ВОЗНИКНОВЕНИЕ ДЕТЕРМИНИРОВАННОГО ХАОСА В НЕКОТОРЫХ НЕИДЕАЛЬНЫХ МАЯТНИКОВЫХ СИСТЕМАХ

Проблематика. Рассматривается влияние запаздывания взаимодействия между маятником и электродвигателем и запаздывания реакции среды на динамику неидеальных маятниковых систем типа "маятник–электродвигатель". Математической моделью такой системы является система обыкновенных дифференциальных уравнений с запаздыванием.

Цель исследования. Цель работы – исследовать влияние факторов запаздывания на установившиеся режимы колебаний неидеальных маятниковых систем типа "маятник–электродвигатель".

Методика реализации. Предложены подходы, которые позволяют свести математическую модель системы к системе трех или пятнадцати дифференциальных уравнений без запаздывания. Для общего анализа нелинейной динамики этих систем построены карты динамических режимов, которые позволяют проводить качественную идентификацию типа установившегося режима. Методика построения карт динамических режимов базируется на анализе спектра ляпуновских характеристических показателей. Построены фазовые портреты регулярных и хаотических аттракторов.

Результаты исследования. Установлено, что для исследования динамики систем "маятник–электродвигатель" при малых значениях запаздывания достаточно использовать трехмерную математическую модель. При сравнительно больших значениях запаздывания необходимо использовать многомерную математическую модель, которая состоит из пятнадцати уравнений.

Выводы. Показано существенное влияние запаздывания на качественное изменение динамических характеристик в системах типа "маятник–электродвигатель". В некоторых случаях запаздывание является управляющим фактором в процессе хаотизации маятниковых систем.

Ключевые слова: маятниковые системы; системы с ограниченным возбуждением; карты динамических режимов; регулярные и хаотические аттракторы; запаздывание.

Рекомендована Радою
фізико-математичного факультету
НТУУ "КПІ"

Надійшла до редакції
27 травня 2015 року